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Linear Stabilization of the Linear Oscillator in Hilbert Space

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1. INTRODUCTION

One of the methods which have been proposed for stabilizing distributed oscillating systems is the following. At a number of stations on the plant are placed sensors to determine the velocity at those stations, together with devices which apply, at those same stations, control forces negatively proportional to the measured velocity. Implementation of this control policy is said to yield quite satisfactory results but, as far as is known to this writer, no rigorous mathematical study has been presented. We shall give such a study in this article.

While our concern lies principally with distributed systems it seems appropriate to introduce our subject with the finite dimensional case. The analysis here is very easy and no originality is claimed.

Consider then a linear oscillator in E^n with r-dimensional controlling force:

$$\frac{d^2y}{dt^2} + Ay = Bu, \tag{1.1}$$

where A is a positive definite symmetric $n \times n$ matrix and B is an $n \times r$ matrix. Our basic assumption is the familiar controllability condition, i.e., that there is a nonnegative integer m such that the rank of the matrix

$$(B, AB, \ldots, A^m B) \tag{1.2}$$

is equal to n.

The mathematical statement of the control policy described in the opening paragraph is the feedback law

$$u = -B^T \frac{dy}{dt}, \tag{1.3}$$

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where the superscript T denotes the transpose. The closed-loop system is then

$$\frac{d^2y}{dt^2} + BB^T \frac{dy}{dt} + Ay = 0. \tag{1.4}$$

We will show that $y = 0$, $(dy/dt) = 0$ is an asymptotically stable critical point.

We define a Liapounov function

$$L\left(y, \frac{dy}{dt}\right) = \frac{1}{2}(y, Ay) + \frac{1}{2}\left(\frac{dy}{dt}, \frac{dy}{dt}\right) \tag{1.5}$$

and compute its derivative along solutions of Eq. (1.4). The result is

$$\frac{d}{dt} L\left(y, \frac{dy}{dt}\right) = -\left(\frac{dy}{dt}, BB^T \frac{dy}{dt}\right) \leqslant 0. \tag{1.6}$$

Moreover, the inequality holds strictly unless

$$B^T \frac{dy}{dt} = 0. \tag{1.7}$$

A result due to La Salle [1] then states that every solution of (1.4) tends to the largest invariant set contained in the set of states y, (dy/dt) obeying (1.7). Thus our work is done if we can show that any solution of (1.4) satisfying (1.7) identically must be the zero solution.

Differentiating (1.7) we have

$$\frac{d}{dt}\left(B^T \frac{dy}{dt}\right) = B^T\left(-BB^T \frac{dy}{dt} - Ay\right) = 0 \tag{1.8}$$

and since (1.7) implies $BB^T(dy/dt) = 0$ we have $B^T Ay = 0$. Differentiating once more yields

$$B^T A \frac{dy}{dt} = 0. \tag{1.9}$$

Continuing this process we have

$$B^T A^\ell \frac{dy}{dt} = 0, \qquad \ell = 0, 1, \ldots, m \tag{1.10}$$

and the assumption made on the matrix (1.2) then implies dy/dt is identically zero. We then return to (1.4) and use the positive definiteness of A to show that y is identically zero. Thus $y = (dy/dt) = 0$ is the only invariant set contained in the set described by (1.7) and we conclude that the control (1.3) stabilizes the system (1.1). By passing to a first order system, we conclude as a corollary that all eigenvalues of the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & I \\ -A & -BB^T \end{pmatrix} \tag{1.11}$$

have negative real parts.

2. Infinite Dimensional Systems: Problem Description and Statement of Results

In this section we will pose an abstract control problem in Hilbert space and state our main theorem. In Section 3 we will discuss both the assumptions and the results from the point of view of applications, the proof of the main theorem being left to Section 4.

Let H denote a complex Hilbert space and A a positive self-adjoint operator, in general unbounded, defined on a domain Δ dense in H. We assume that the spectrum of A consists of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \tag{2.1}$$

of single multiplicity, with corresponding eigenvectors $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ forming a complete orthonormal set in H. Taking

$$\omega_k = \begin{cases} \lambda_k^{1/2}, & k = 1, 2, \ldots, \\ -\lambda_{-k}^{1/2}, & k = -1, -2, \ldots, \end{cases} \tag{2.2}$$

we make the assumption

$$\frac{\omega_k}{\lambda_k - \lambda_{k-1}} \leqslant M, \qquad k = 2, 3, \ldots, \qquad M > 0. \tag{2.3}$$

For later convenience we set

$$\phi_k = \phi_{-k}, \qquad \lambda_k = \lambda_{-k}, \qquad k = -1, -2, \ldots . \tag{2.4}$$

Taking g to be a vector in H we consider a linear oscillator in H with scalar control u:

$$\frac{d^2y}{dt^2} + Ay = gu. \tag{2.5}$$

The condition on g analogous to that imposed on the matrix B in Section 1 is this: Let the expansion of g in terms of the ϕ_k be

$$g = \sum_{k=1}^{\infty} \gamma_k \phi_k . \tag{2.6}$$

(As in (2.4), set $\gamma_k = \gamma_{-k}$, $k = -1, -2, \ldots$). Then we require that

$$0 < |\gamma_k| \leqslant \hat{M} \frac{1}{\omega_k}, \qquad k = 1, 2, \ldots, \qquad \hat{M} > 0. \tag{2.7}$$

The assumption expressed by the second inequality can probably be greatly weakened but is needed here because we use a perturbation technique.

Let us consider the feedback control law

$$u = -\mathscr{E}\left(\frac{dy}{dt}, g\right), \qquad \mathscr{E} > 0. \tag{2.8}$$

Defining a linear operator G on H by

$$Gy = (y, g)\, g \tag{2.9}$$

the closed loop system becomes

$$\frac{d^2y}{dt^2} + \mathscr{E}G\frac{dy}{dt} + Ay = 0. \tag{2.10}$$

Setting

$$y = A^{-1/2}y^1, \qquad \frac{dy}{dt} = y^2, \tag{2.11}$$

where $A^{1/2}$ is the unique positive square root of A, we obtain a first-order system

$$\frac{d}{dt}\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -\mathscr{E}G \end{pmatrix}\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \tag{2.12}$$

in the Hilbert space $H \oplus H$. We should emphasize that Eqs. (2.10) and (2.12) are not strictly equivalent—a solution of one represents a solution of the other only in a generalized sense unless special conditions are assumed on G and the initial values of the solution of (2.12).

We define a solution of (2.12) to be a vector function of the form

$$\exp\left[\begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -\mathscr{E}G \end{pmatrix} t\right]\begin{pmatrix} y_0^1 \\ y_0^2 \end{pmatrix}, \qquad \begin{pmatrix} y_0^1 \\ y_0^2 \end{pmatrix} \in H \oplus H. \tag{2.13}$$

Clearly the properties of such solutions will depend upon the nature of the spectrum of the operator

$$\tilde{A}(\mathscr{E}) = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -\mathscr{E}G \end{pmatrix}.$$

$\tilde{A}(\mathscr{E})$ is defined on $\Delta_1 \oplus \Delta_1$, where $\Delta_1 \supseteq \Delta$ is the domain of $A^{1/2}$.

Our main theorem is the following:

THEOREM 1. *If $\mathscr{E} > 0$ is sufficiently small, the operator $\tilde{A}(\mathscr{E})$ has eigenvalues $\nu_k(\mathscr{E})$ all with negative real parts, more precisely*

$$\nu_k(\mathscr{E}) = i\omega_k - \frac{\mathscr{E}}{2} \mid \gamma_k \mid^2 + 0 \left(\mathscr{E}^2 \frac{1}{\mid \omega_k \mid^2} \right), \qquad k = \pm 1, \pm 2, \ldots, \tag{2.14}$$

and corresponding eigenvectors $\psi_k(\mathscr{E})$, $k = \pm 1, \pm 2, \ldots$ forming a Riesz basis for $H \oplus H$, i.e. given $z \in H \oplus H$,

$$z = \sum_{k=1}^{\infty} (\zeta_k(\mathscr{E})\, \psi_k(\mathscr{E}) + \zeta_{-k}(\mathscr{E})\, \psi_{-k}(\mathscr{E})) \tag{2.15}$$

and, for positive numbers m and M,

$$m \sum_{k=1}^{\infty} (\mid \zeta_k \mid^2 + \mid \zeta_{-k} \mid^2) \leqslant \| z \|^2 \leqslant M \sum_{k=1}^{\infty} (\mid \zeta_k \mid^2 + \mid \zeta_{-k} \mid^2). \tag{2.16}$$

Thus $\exp(\tilde{A}(\mathscr{E})\, t)$ *tends strongly to zero as $t \to \infty$ and every solution* (2.13) *of* (2.12) *approaches zero as $t \to \infty$.*

3. Remarks and Applications

The results of the theorem stated above are certainly not entirely satisfactory. Intuitively it seems clear that the result should be true for all positive $\mathscr{E}$, indeed an increase in $\mathscr{E}$ should improve the stability of the system. Such a global result is not obtained as our proof is based on perturbation theory of linear operators rather than Liapounov theory. Although a Liapounov theory does exist for differential equations in Hilbert space the writer is unaware of any extension of LaSalle's theorem [1] to infinite dimensional spaces. This is due to the fact that the arguments concerning invariant sets, etc., rely heavily upon the compactness of closed bounded sets—true in E^n but not in Hilbert spaces of infinite dimension. Although the use of perturbation theory carries with it the disadvantage that $\mathscr{E}$ must be small, this technique offers many advantages as well. It seems very doubtful that the precision of the results expressed in the above theorem could be realized by any other technique. We will make use of this precise information very shortly.

The assumptions expressed by (2.1) and (2.3) may seem rather restrictive at first and this is partly true. Essentially, the assumption (2.3) restricts application to partial differential equations in one space variable. Again we note that assumption (2.3) is made so that the perturbation technique will work. This assumption would be rather unnatural in a Liapounov theory setting. However, given that we are dealing with one space variable, (2.3) is not unduly restrictive. The equations

$$\rho(x)\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial y}{\partial x}\right) = g(x)\,u(t) \tag{3.1}$$

and

$$\rho(x)\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(p(x)\frac{\partial^2 y}{\partial x^2}\right) = g(x)\,u(t) \tag{3.2}$$

for the vibrating string and simple beam, respectively, give rise to operators

$$Ay = \frac{-1}{\rho(x)}\frac{\partial}{\partial x}\left(p(x)\frac{\partial y}{\partial x}\right) \qquad \text{and} \qquad Ay = \frac{1}{\rho(x)}\frac{\partial^2}{\partial x^2}\left(p(x)\frac{\partial^2 y}{\partial x^2}\right)$$

which are positive and self-adjoint, provided appropriate boundary conditions are imposed. By using well-known estimates for the eigenvalues λ_k (see e.g., [2] in the case of (3.1), [3] in the case of (3.2)) the condition (2.3) is readily seen to hold in these cases.

In Kalman's theory of optimal control with a quadratic cost criterion (see, e.g., [4]) the proof of the existence of an optimal linear control relative to a cost functional involving an integral over the infinite time interval depends upon the assumption that there is a linear control yielding finite cost. In another article [8] it is shown that similar considerations apply to linear control systems in Hilbert space. We will indicate here that Theorem 1 can be used to advantage in such a study.

THEOREM 2. *Let a cost functional $C(u)$ be defined by*

$$C(u) = \int_0^\infty \left[(y(t), y(t)) + \left(A^{-1/2}\frac{dy}{dt}, A^{-1/2}\frac{dy}{dt}\right) + |\,u(t)\,|^2\right] dt, \tag{3.3}$$

where $y(t)$ and $u(t)$ satisfy (2.5). *If there is a positive number L such that, in addition to* (2.7)

$$|\,\gamma_k\,| \geqslant \frac{L}{\omega_k}, \qquad k = 1, 2, \ldots \tag{3.4}$$

and if the initial energy

$$\frac{1}{2}\,(A^{1/2}y(0), A^{1/2}y(0)) + \frac{1}{2}\left(\frac{dy}{dt}(0), \frac{dy}{dt}(0)\right)$$

is finite, i.e., if

$$y(0) = A^{-1/2}y^1(0), \qquad y^1(0) \in H, \tag{3.5}$$

then the cost (3.3) *associated with the control* (2.8) *is finite.*

PROOF. Since the energy associated with a solution of (2.10) is non-increasing, $A^{1/2}y(t)$ must be defined for all t. The transformation (2.11) is then valid and we may pass to the system (2.12). In terms of $y^1(t)$ and $y^2(t)$ we have

$$C(u) = \int_0^\infty [(A^{-1/2}y^1(t), A^{-1/2}y^1(t)) + (A^{-1/2}y^2(t), A^{-1/2}y^2(t)) + |u(t)|^2]\, dt. \tag{3.6}$$

Now

$$\int_0^\infty |u(t)|^2\, dt \leqslant \mathscr{E}E, \tag{3.7}$$

where E is the initial system energy. This is clear because the difference between the energy at a time $T > 0$ and the energy at time 0 is

$$\begin{aligned} -\int_0^T (y^2(t), \mathscr{E}Gy^2(t))\, dt &= -\int_0^T \mathscr{E}(y^2(t), (y^2(t), g)\, g)\, dt \\ &= -\frac{1}{\mathscr{E}}\int_0^T |\mathscr{E}(y_2(t), g)|^2\, dt = -\frac{1}{\mathscr{E}}\int_0^T |u(t)|^2\, dt \end{aligned} \tag{3.8}$$

and since the first integral is bounded by E as $T \to \infty$, we have (3.7). Thus we need only study

$$\int_0^\infty \| A_1^{-1}\eta(t) \|^2\, dt, \tag{3.9}$$

where

$$\eta(t) = \begin{pmatrix} y^1(t) \\ y^2(t) \end{pmatrix}, \qquad A_1 = \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix}.$$

We claim now that there is a constant $K > 0$ such that

$$\| A_1^{-1}\eta \| \leqslant K \| \tilde{A}(\mathscr{E})^{-1}\, \eta \|, \tag{3.10}$$

$\tilde{A}(\mathscr{E})$ being given in (2.13). We remark that Theorem 1 shows that $\tilde{A}(\mathscr{E})$ is invertible if $\mathscr{E}$ is sufficiently small. Let

$$\eta = \tilde{A}(\mathscr{E})_\eta \tag{3.11}$$

and compute

$$\begin{aligned} \| A_1^{-1}\tilde{A}(\mathscr{E})\, \hat{\eta} \| &= \left\| \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -\mathscr{E}G \end{pmatrix} \begin{pmatrix} \hat{y}^1 \\ \hat{y}^2 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} 0 & I \\ -I & -\mathscr{E}A^{-\frac{1}{2}}G \end{pmatrix} \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} \right\| \leqslant K \left\| \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} \right\| = K \| \hat{\eta} \|. \end{aligned} \tag{3.12}$$

Setting $\hat{\eta} = \tilde{A}(\mathscr{E})^{-1}\, \eta$ in (3.12) we have (3.10).

It now suffices to show that

$$\int_0^\infty \| \tilde{A}(\mathscr{E})^{-1} \eta(t) \|^2 \, dt < \infty. \tag{3.13}$$

But, assuming $\quad \eta(0) = \sum_{k=1}^{\infty} (\eta_k(\mathscr{E}) \, \psi_k(\mathscr{E}) + \eta_{-k}(\mathscr{E}) \, \psi_{-k}(\mathscr{E})),$

$$\begin{aligned} \tilde{A}(\mathscr{E})^{-1} \eta(t) &= \tilde{A}(\mathscr{E})^{-1} \exp(\tilde{A}(\mathscr{E}) \, t) \, \eta(0) \\ &= \sum_{k=1}^{\infty} \left(\frac{\eta_k(\mathscr{E})}{\nu_k(\mathscr{E})} \exp(\nu_k(\mathscr{E}) \, t) \, \psi_k(\mathscr{E}) + \frac{\eta_{-k}(\mathscr{E})}{\nu_{-k}(\mathscr{E})} \exp(\nu_{-k}(\mathscr{E}) \, t) \, \psi_{-k}(\mathscr{E}) \right) \end{aligned} \tag{3.14}$$

and (2.16) then yields

$$\begin{aligned} \int_0^\infty \| \tilde{A}(\mathscr{E})^{-1} \eta(t) \|^2 \, dt \leqslant M \sum_{k=1}^{\infty} \Bigg(\left| \frac{\eta_k(\mathscr{E})}{\nu_k(\mathscr{E})} \right|^2 \int_0^\infty | \exp(2\nu_k(\mathscr{E}) \, t) | \, dt \\ + \left| \frac{\eta_{-k}(\mathscr{E})}{\nu_{-k}(\mathscr{E})} \right|^2 \int_0^\infty | \exp(2\nu_{-k}(\mathscr{E}) \, t) | \, dt \Bigg). \end{aligned} \tag{3.15}$$

Now (2.14) and (3.4) imply that

$$\operatorname{Re}(2\nu_k(\mathscr{E})) \leqslant \frac{-\mathscr{E}L^2}{\omega_k^{\,2}} \tag{3.16}$$

if $\mathscr{E}$ is sufficiently small. Also for small $\mathscr{E}$ we have

$$| \nu_k(\mathscr{E}) | \geqslant a\omega_k^{\,2} \tag{3.17}$$

for some $a > 0$. Hence

$$\left| \frac{\eta_k(\mathscr{E})}{\nu_k(\mathscr{E})} \right|^2 \int_0^\infty | \exp(2\nu_k(\mathscr{E}) \, t) | \, dt \leqslant \frac{| \eta_k(\mathscr{E}) |^2}{a\omega_k^{\,2}} \frac{\omega_k^{\,2}}{\mathscr{E}L^2} = \frac{| \eta_k(\mathscr{E}) |^2}{\mathscr{E}aL^2}. \tag{3.18}$$

Thus we see that

$$\int_0^\infty \| \tilde{A}(\mathscr{E})^{-1} \eta(t) \|^2 \, dt \leqslant \frac{M}{\mathscr{E}aL^2} \sum_{k=1}^{\infty} (| \eta_k(\mathscr{E}) |^2 + | \eta_{-k}(\mathscr{E}) |^2). \tag{3.19}$$

Combining this result with (3.7),

$$C(u) \leqslant \frac{Mm}{\mathscr{E}aL^2} \| \eta(0) \|^2 + \mathscr{E}E, \tag{3.20}$$

where E is the initial energy. With this the proof of Theorem 2 is complete.

It is interesting that assumption (3.4) is the same as the assumption made in [5] to guarantee finite time controllability of the vibrating string with a square integrable control.

Theorem 2 may also be viewed as a sort of estimate on the rate of decrease of $\| y(t) \|$. Loosely speaking, we would expect $\| y(t) \| < Ct^{-1/2}$ as $t \to \infty$, although this is not logically a consequence of the theorem.

4. Proof of Theorem 1

Let ν be a complex number and consider the identity

$$\begin{pmatrix} -\nu I & A^{1/2} \\ -A^{1/2} & -\nu I - \mathscr{E} G \end{pmatrix} \begin{pmatrix} A^{1/2}\phi \\ \nu\phi \end{pmatrix} = \begin{pmatrix} 0 \\ -(A + \mathscr{E}\nu G + \nu^2 I)\phi \end{pmatrix} \tag{4.1}$$

valid for

$$\phi \in \Delta = \mathrm{dom}(A). \tag{4.2}$$

If there exists a nonzero $\phi \in \Delta$ such that

$$(A + \mathscr{E}\nu G + \nu^2 I)\phi = 0 \tag{4.3}$$

then ν is an eigenvalue of $\tilde{A}(\mathscr{E})$ and

$$\psi = c \begin{pmatrix} A^{1/2}\phi \\ \nu\phi \end{pmatrix}, \tag{4.4}$$

is an associated eigenvector, c being an arbitrary nonzero scalar. When $\mathscr{E} = 0$, if we take

$$\nu = i\omega_k, \qquad \phi = \phi_k, \qquad c = \frac{1}{(2)^{1/2} \mid \omega_k \mid}, \qquad k = \pm 1, \pm 2, \ldots, \tag{4.5}$$

we obtain the eigenvalues and normalized eigenvectors of the operator $\tilde{A}(0)$. It is our intention to use (4.3) to find eigenvalues and eigenvectors of $\tilde{A}(\mathscr{E})$ reducing to the above-mentioned eigenvalues and eigenvectors of $\tilde{A}(0)$ as $\mathscr{E} \to 0$.

For $k = \pm 1, \pm 2, \ldots$ let us set

$$\nu_k(\mathscr{E}) = i\omega_k - \frac{\mathscr{E}}{2} \mid \gamma_k \mid^2 + \mathscr{E}\mu_k(\mathscr{E}), \tag{4.6}$$

$$\phi_k(\mathscr{E}) = \phi_k - \mathscr{E} i\omega_k (A_k - \lambda_k E_k)^{-1} E_k G \phi_k + \mathscr{E}(A_k - \lambda_k E_k)^{-1} \theta_k(\mathscr{E}). \tag{4.7}$$

The operator E_k is the orthogonal projection from H onto H_k, which consists of vectors orthogonal to ϕ_k. A_k is the restriction of A to H_k. Our hypotheses on A guarantee that λ_k is not in the spectrum of A_k, hence $(A_k - \lambda_k E_k)^{-1}$ is defined as a bounded operator on H_k, It is assumed that $\theta_k(\mathscr{E}) \in H_k$.

We substitute (4.6) and (4.7) in place of ν and ϕ in the Eq. (4.3). After a little manipulation we arrive at the equation

$$
\begin{aligned}
(A - \lambda_k I)\phi_k &+ \mathscr{E}\{[i\omega_k G - i\omega_k \mid \gamma_k \mid^2 I + 2i\omega_k \mu_k(\mathscr{E}) I]\phi_k \\
&\qquad + [-i\omega_k E_k G\phi_k + \theta_k(\mathscr{E})]\} \\
&+ \mathscr{E}^2\{[-\tfrac{1}{2} \mid \gamma_k \mid^2 G + \mu_k(\mathscr{E}) G + (-\tfrac{1}{2} \mid \gamma_k \mid^2 + \mu_k(\mathscr{E}))^2]\phi_k \\
&\qquad + [i\omega_k G - i\omega_k \mid \gamma_k \mid^2 I + 2i\omega_k \mu_k(\mathscr{E}) I] \\
&\qquad \times [(A_k - \lambda_k E_k)^{-1}(-i\omega_k E_k G\phi_k + \theta_k(\mathscr{E}))]\} \\
&+ \mathscr{E}^3[-\tfrac{1}{2} \mid \gamma_k \mid^2 G + \mu_k(\mathscr{E}) G + (-\tfrac{1}{2} \mid \gamma_k \mid^2 + \mu_k(\mathscr{E}))^2 I)] \\
&\qquad \times [(A_k - \lambda_k E_k)^{-1}(-i\omega_k E_k G\phi_k + \theta_k(\mathscr{E}))] = 0. \qquad (4.8)
\end{aligned}
$$

Let us denote the left-hand side of (4.8) by $x_k(\mathscr{E}) \in H$. We have $x_k(\mathscr{E}) = 0$ if and only if

$$(x_k(\mathscr{E}), \phi_k) = 0, \qquad (4.9)$$

$$E_k x_k(\mathscr{E}) = 0. \qquad (4.10)$$

If we use the fact that $(A - \lambda_k I)\phi_k = 0$, $(G\phi_k, \phi_k) = \mid \gamma_k \mid^2$, $E_k\phi_k = 0$, the equations (4.9) and (4.10) become, after division by $2\mathscr{E} i\omega_k$, $\mathscr{E}$, respectively,

$$
\begin{aligned}
\mu_k(\mathscr{E}) &+ \frac{\mathscr{E}}{2i\omega_k}\Big\{\Big[-\frac{1}{4} \mid \gamma_k \mid^4 + 2\mu_k(\mathscr{E}) \mid \gamma_k \mid^2 + \mu_k(\mathscr{E})^2\Big] \\
&\qquad + (-i\omega_k G(A_k - \lambda_k E_k)^{-1}(-i\omega_k E_k G\phi_k + \theta_k(\mathscr{E})), \phi_k)\Big\} \\
&+ \frac{\mathscr{E}^2}{2i\omega_k}\Big(\Big[\Big(-\frac{1}{2} \mid \gamma_k \mid^2 + \mu_k(\mathscr{E})\Big) G\Big] \\
&\qquad \times [(A_k - \lambda_k E_k)^{-1}(-i\omega_k E_k G\phi_k + \theta_k(\mathscr{E}))], \phi_k\Big) = 0, \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
\theta_k(\mathscr{E}) &+ \mathscr{E}\{(-\tfrac{1}{2} \mid \gamma_k \mid^2 + \mu_k(\mathscr{E}))E_k G\phi_k \\
&\qquad + [i\omega_k E_k G - i\omega_k \mid \gamma_k \mid^2 E_k + 2i\omega_k \mu_k(\mathscr{E}) E_k] \\
&\qquad \times [(A_k - \lambda_k E_k)^{-1}(-i\omega_k E_k G\phi_k + \theta_k(\mathscr{E}))]\} \\
&+ \mathscr{E}^2[-\tfrac{1}{2} \mid \gamma_k \mid^2 E_k G + \mu_k(\mathscr{E}) E_k G + (-\tfrac{1}{2} \mid \gamma_k \mid^2 + \mu_k(\mathscr{E}))^2 E_k] \\
&\qquad \times [(A_k - \lambda_k E_k)^{-1}(-i\omega_k E_k G\phi_k + \theta_k(\mathscr{E}))] = 0. \qquad (4.12)
\end{aligned}
$$

Now let us observe that (2.3) implies

$$\| (A_k - \lambda_k E_k)^{-1} \| \leqslant \frac{M}{\mid \omega_k \mid} \qquad (4.13)$$

while

$$\| G\phi_k \| = \| (\phi_k , g) g \| = | \gamma_k | \, \| g \| . \tag{4.14}$$

If (4.11) and (4.12) are examined in detail, using (4.13), (4.14), and (2.7) together with the self-adjointness of G, they are seen to have the form

$$\mu_k(\mathscr{E}) \, [1 + \mathscr{E} F_k(\mathscr{E}, \mu_k(\mathscr{E}), \theta_k(\mathscr{E}))] = \mathscr{E} G_k(\mathscr{E}, \theta_k(\mathscr{E})) \tag{4.15}$$

$$\theta_k(\mathscr{E}) + \mathscr{E} H_k(\mathscr{E}, \mu_k(\mathscr{E}), \theta_k) = \mathscr{E} J_k(\mathscr{E}, \mu_k(\mathscr{E})) \tag{4.16}$$

where

$$| F_k(\mathscr{E}, \mu_k(\mathscr{E}), \theta_k(\mathscr{E})) | \leqslant C_1(1 + | \mu_k(\mathscr{E}) | + \| \theta_k(\mathscr{E}) \|) \tag{4.17}$$

$$| G_k(\mathscr{E}, \theta_k(\mathscr{E})) | \leqslant \frac{C_2}{| \omega_k |^2} (1 + \| \theta_k(\mathscr{E}) \|) \tag{4.18}$$

$$\| H_k(\mathscr{E}, \mu_k(\mathscr{E}), \theta_k(\mathscr{E})) \| \leqslant C_3(1 + | \mu_k(\mathscr{E}) | + \| \theta_k(\mathscr{E}) \|) \tag{4.19}$$

$$\| J_k(\mathscr{E}, \mu_k(\mathscr{E})) \| \leqslant C_4(1 + | \mu_k(\mathscr{E}) |), \tag{4.20}$$

where C_1, C_2, C_3, C_4 are positive constants which do not depend upon k. In addition, the Fréchet partial derivatives of $H_k(\mathscr{E}, \mu_k(\mathscr{E}), \theta_k(\mathscr{E}))$ and $J_k(\mathscr{E}, \mu_k(\mathscr{E}))$ with respect to $\mu_k(\mathscr{E})$ and $\theta_k(\mathscr{E})$ can be uniformly bounded in terms of $| \mu_k(\mathscr{E}) |$ and $\| \theta_k(\mathscr{E}) \|$, independent of k. One may now apply the implicit function theorem in its general form (see, e.g., [6]), which is valid for functions on complete normed linear spaces, to see that (4.15) and (4.16) have solutions $\mu_k(\mathscr{E})$, $\theta_k(\mathscr{E})$, satisfying

$$| \mu_k(\mathscr{E}) | \leqslant K_1 | \mathscr{E} | \frac{1}{| \omega_k |^2} \tag{4.21}$$

$$\| \theta_k(\mathscr{E}) \| \leqslant K_2 | \mathscr{E} | \tag{4.22}$$

defined for

$$| \mathscr{E} | \leqslant \mathscr{E}_0 , \tag{4.23}$$

where K_1, K_2 and $\mathscr{E}_0$ are positive numbers which are independent of k. Going back to (4.6) and (4.7), we see that (4.3) has solutions

$$\nu_k(\mathscr{E}) = i\omega_k - \frac{\mathscr{E}}{2} | \gamma_k |^2 + 0 \left(| \mathscr{E} |^2 \frac{1}{| \omega_k |^2} \right) . \tag{4.24}$$

$$\phi_k(\mathscr{E}) = \phi_k - \mathscr{E} i\omega_k (A_k - \lambda_k E_k)^{-1} E_k G \phi_k + \mathscr{E} (A_k - \lambda_k E_k)^{-1} \theta_k(\mathscr{E}) \tag{4.25}$$

valid for $| \mathscr{E} | \leqslant \mathscr{E}_0$. Note that (4.24) is exactly the statement (2.14) of Theorem 1.

Let us now set

$$\psi_k(\mathscr{E}) = \frac{1}{(2)^{1/2} \mid \omega_k \mid} \begin{pmatrix} A^{1/2} & \phi_k(\mathscr{E}) \\ \nu_k(\mathscr{E}) & \phi_k(\mathscr{E}) \end{pmatrix} = \frac{1}{(2)^{1/2} \mid \omega_k \mid} \begin{pmatrix} \mid \omega_k \mid \phi_k \\ i\omega_k \phi_k \end{pmatrix} + \frac{1}{(2)^{1/2} \mid \omega_k \mid}$$

$$\times \left\{ \begin{array}{l} -\mathscr{E}[A^{1/2}(A_k - \lambda_k E_k)^{-1} (i\omega_k E_k G\phi_k + \theta_k(\mathscr{E}))] \\ \left\{ \begin{array}{l} \mathscr{E}[- i\omega_k(A_k - \lambda_k E_k)^{-1} (i\omega_k E_k G\phi_k + \theta_k(\mathscr{E})) \\ \quad + \left(-\frac{1}{2} \mid \gamma_k \mid^2 + 0\left(\mid \mathscr{E} \mid \frac{1}{\mid \omega_k \mid^2}\right)\right) \phi_k\Big) \\ \quad + \mathscr{E}^2 \left[-\frac{1}{2} \mid \gamma_k \mid^2 + 0\left(\mid \mathscr{E} \mid \frac{1}{\mid \omega_k \mid^2}\right)\right] \\ \quad \times [(A_k - \lambda_k E_k)^{-1} (i\omega_k E_k G\phi_k + \theta_k(\mathscr{E}))] \end{array} \right. \end{array} \right\}. \tag{4.26}$$

Let P_k denote the projection $I - E_k$. The spectral theory of the self-adjoint operator A shows that

$$A^{1/2}(A_k - \lambda_k E_k)^{-1} = \sum_{\ell \neq k} \frac{\omega_k}{\lambda_\ell - \lambda_k} P_\ell. \tag{4.27}$$

Now the assumption (2.3) implies that

$$\left| \frac{\omega_k}{\lambda_\ell - \lambda_k} \right| \leqslant M, \qquad \ell \neq k, \tag{4.28}$$

and hence that

$$\| A^{1/2}(A_k - \lambda_k E_k)^{-1} \| \leqslant M. \tag{4.29}$$

Combining (4.29) with (4.14) and (2.7) we see that (4.26) becomes

$$\psi_k(\mathscr{E}) = \psi_k(0) + 0\left(\mid \mathscr{E} \mid \frac{1}{\mid \omega_k \mid}\right), \tag{4.30}$$

where the

$$\psi_k(0) = \frac{1}{(2)^{1/2} \mid \omega_k \mid} \begin{pmatrix} \mid \omega_k \mid \phi_k \\ i\omega_k \phi_k \end{pmatrix}$$

form a complete orthonormal set in $H \oplus H$. Now (2.7) also implies that

$$\frac{1}{\mid \omega_k \mid} = 0\left(\frac{1}{\mid k \mid}\right), \qquad k = \pm 1, \pm 2, \ldots \tag{4.31}$$

so that the terms denoted $0(\mid \mathscr{E} \mid 1/\mid \omega_k \mid)$ in (4.30) are such that the sequence composed of the squares of the norms of these terms is summable. A theorem of Paley and Wiener (see statement and proof in [7], e.g.) then shows that the $\psi_k(\mathscr{E})$ have the property of being a Riesz basis as stated in Theorem 1.

Since in Theorem 1 we have not made any assumption of the type (3.4) the result (4.24) does not, by itself, imply that $\mathrm{Re}(\nu_k(\mathscr{E})) < 0$. That this is true is proved as follows. We let ϕ be a vector of unit norm and ν a complex number which together satisfy (4.3). Then

$$0 = (\phi, (A + \mathscr{E}\nu G + \nu^2 I)\phi) = (\phi, A\phi) + \mathscr{E}\nu(\phi, G\phi) + \nu^2 \qquad (4.32)$$

which is a quadratic equation in ν. Thus

$$\nu = \tfrac{1}{2}\{-\mathscr{E}(\phi, G\phi) \pm [\mathscr{E}^2(\phi, G\phi)^2 - 4(\phi, A\phi)]^{1/2}\}.$$

Since A is positive and bounded away from 0, for small values of $\mathscr{E}$ the term $\mathscr{E}^2(\phi, G\phi)^2 - 4(\phi, A\phi)$ is nonpositive. The smallness condition on $\mathscr{E}$ is independent of ϕ. Thus

$$\mathrm{Re}(\nu) = -\frac{\mathscr{E}}{2}(\phi, G\phi) \leqslant 0 \qquad (4.33)$$

and equality holds only if (g, ϕ), and hence $G\phi$, is zero. But if $G\phi = 0$, (4.3) shows ϕ to be an eigenvector of A. So $\phi = \phi_k$ for some k. But then (2.6) shows that

$$\gamma_k = (g, \phi_k) = 0$$

which contradicts (2.7). Hence the inequality in (4.33) is strict and the proof of Theorem 1 is complete.

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